Algebraic Geometry, Part II, Example Sheet 4,2019

Assume throughout that the base field k is algebraically closed. If it helps, feel free to assume throughout that it has characteristic zero.

1. A smooth irreducible projective curve V is covered by two affine pieces (with respect to different embeddings) which are affine plane curves with equations $y^2 = f(x)$ and $v^2 = g(u)$ respectively, with f a square-free polynomial of even degree $2n$ and $u = 1/x$, $v = y/x^n$ in $k(V)$. Determine the polynomial $g(u)$ and show that the canonical class on V has degree $2n - 4$. Why can we not just say that V is the projective plane curve associated to the affine curve $y^2 = f(x)$?

2. Let $V_0 \subset \mathbb{A}^2$ be the affine curve with equation $y^3 = x^4 + 1$, and let $V \subset \mathbb{P}^2$ be its projective closure. Show that V is smooth, and has a unique point Q at infinity. Let ω be the rational differential dx/y^2 on V. Show that $v_P(\omega) = 0$ for all $P \in V_0$. prove that $v_Q(\omega) = 4$ and hence that ω, $x\omega$ and $y\omega$ are all regular on V.

3. Let V be a smooth irreducible projective curve and $P \in V$ any point. Show that there exists a nonconstant rational function on V which is regular everywhere except at P. Show moreover that there exists an embedding $\phi\colon V \hookrightarrow \mathbb{P}^n$ such that $\phi^{-1}(\{X_0 = 0\}) = \{P\}$. In particular, $V \setminus \{P\}$ is an affine curve. If V has genus g, show that there exists a nonconstant morphism $V \to \mathbb{P}^1$ of degree g.

4. Let P_∞ be a point on an elliptic curve X (smooth irreducible projective curve of genus 1) and $\alpha_{3P_\infty} : X \xrightarrow{\sim} W \subset \mathbb{P}^2$ the projective embedding, with image W. Show that $P \in W$ is a point of inflection if and only if $3P = 0$ in the group law determined by P_∞. Deduce that if P and Q are points of inflection then so is the third point of intersection of the line PQ with W.

5. Let $V:\ ZY^2 + Z^2Y = X^3 - XZ^2$ and take $P_0 = (0:1:0)$ for the identity of the group law. Calculate the multiples $nP = P \oplus \cdots \oplus P$ of $P = (0:0:1)$ for $2 \le n \le 4$.

6. Show that any morphism from a smooth irreducible projective curve of genus 4 to a smooth irreducible projective curve of genus 3 must be constant.

7. (Assume $\mathrm{char}(k) \neq 2$) (i) Let $\pi\colon V \to \mathbb{P}^1$ be a hyperelliptic cover, and $P \neq Q$ ramification points of π. Show that $P - Q \not\sim 0$ but $2(P - Q) \sim 0$.

   (ii) Let $g(V) = 2$. Show that every divisor of degree 2 on V is linearly equivalent to $P + Q$ for some $P, Q \in V$, and deduce that every divisor of degree 0 is linearly equivalent to $P - Q'$ for some $P, Q' \in V$.

   (iii) Show that if $g(V) = 2$ then the subgroup $\{\,[D] \in \mathrm{Cl}^0(V) \mid 2[D] = 0\,\}$ of the divisor class group of V has order 16.

8. Show that a smooth plane quartic is never hyperelliptic.

9. Let $V:\ X_0^6 + X_1^6 + X_2^6 = 0$, a smooth irreducible plane curve. By applying the Riemann–Hurwitz formula to the projection to $\mathbb{P}^1$ given by $(X_0 : X_1)$, calculate the genus of V.

   Now let $\phi\colon V \to \mathbb{P}^2$ be the morphism $(X_i) \mapsto (X_i^2)$. Identify the image of ϕ and compute the degree of ϕ.

10. Let $V \subset \mathbb{P}^3$ be the intersection of the quadrics $Z(F)$, $Z(G)$ where $\mathrm{char}(k) = 0$ and

$$F = X_0X_1 + X_2^2, \quad G = \sum_{i=0}^{3} X_i^2$$

   (i) Show that V is a smooth curve (possibly reducible).

   (ii) Let $\phi = (X_0\, X_1\, X_2)\colon \mathbb{P}^3 \dashrightarrow \mathbb{P}^2$. (This map is the projection from the point $(\,0\,0\,0\,1)$ to $\mathbb{P}^2$.) Show that $\phi(V)$ is a conic $C \subset \mathbb{P}^2$. By parametrising C, compute the ramification of ϕ and show that $\phi\colon V \to C$ has degree 2. Deduce that V is irreducible of genus 1.